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Technical Notes

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Linear Instability of Laterally Strained Constant Pressure Boundary-Layer Flows

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I. Introduction

THE linear instability of many two-dimensional flows are quite well understood (for example, see Drazin and Reid[1]). Generally, flows are associated with lateral steamline divergence/convergence, longitudinal streamline curvature, etc. This Note is concerned with the linear instability of laterally diverging/converging constant pressure boundary layer flows. Streamline divergence/convergence adds an additional lateral strain rate to a nominally two-dimensional flow. The effect of such an additional strain rate on the constant pressure turbulent boundary layer has been studied by Saddoughi and Joubert[2] and Panchapakesan et al.[3] and on the constant pressure transitional flow by Ramesh[4] and Vasudevan.[5] The linear instability of a constant pressure diverging/converging flow, however, has not been reported in the available literature and has led to some difficulties in the study of the transitional flow characteristics of these flows[5]; an understanding of the effect of lateral strain rate alone on the linear instability surely is desirable. Following Kehl (see Ref. 6), a streamline diverging/converging flow can be approximated by a source/sink flow and, under this approximation, it differs from the usual two-dimensional flow by a source term in the continuity equation. Recently, Ramesh et al.[7] have shown that, under the source/sink flow approximation, a streamline diverging/converging boundary flow can be reduced to a two-dimensional flow by means of the Mangler transformation, discussed in the sequel. The laminar flow measurements have confirmed[4,5] that the velocity profile is Blasius, but the boundary layer is either thicker (for converging flow) or thinner (for diverging flow) than the Blasius flow. The Mangler transformation invovles a nonlinear coordinate transformation, thus rendering it difficult to visualize the instability in the physical coordinates. In this Note, the linear instability of a converging/diverging boundary-layer flow is examined by directly reducing the flow to two-dimensional flow.

II. Analysis and Discussion

A. Mean Flow

Consider a laterally diverging/converging flow in the streamwise direction x, with a freestream velocity u_∞, as shown in Fig. 1; z is the spanwise direction and y is normal to the x–z plane. The velocity components in x, y, and z directions are u, v, and w, respectively. Following Khel (see Ref. 6), a diverging flow is assumed to emanate from a source at $x = -A$, and a converging flow is assumed to have a sink at $x = A$ (Fig. 1). The spanwise velocity is taken as[6] $w = uz/(A + x)$. The quantity $(A + x) > 0$ for a diverging flow, and $(A + x) < 0$ for a converging flow (because A is negative). Along $z = 0$, $w = 0$, but $\partial w/\partial z \neq 0$. This nonzero gradient appears as a source/sink term in the continuity equation[6]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{A + x} = 0 \tag{1}$$

The boundary-layer momentum equation is[6]

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_\infty \frac{\mathrm{d}u_\infty}{\mathrm{d}x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

where ν is the kinematic viscosity. The boundary conditions are

$$u = v = 0, \qquad \text{at } y = 0, \qquad u \to u_\infty, \qquad \text{as } y \to \infty$$

Momentum equation (2) is two dimensional, but continuity equation (1) has an additional source/sink term, compared to two-dimensional flows. Ramesh et al.[7] have used the Mangler-type transformation

$$X = \frac{(A + x)^3}{3A^2}, \qquad Y = \frac{y(A + x)}{A}$$

$$u = U_1, \qquad V_1 = \frac{A}{A + x}\left[v + \frac{yu}{A + x}\right] \tag{3}$$

to reduce Eqs. (1) and (2) to two-dimensional equations. Because this transformation involves a nonlinear coordinate transformation in x, it makes it difficult to study the usual normal mode instability for two-dimensional flows. We propose the following alternative reduction of Eqs. (1) and (2) to two-dimensional flow, and then proceed to study the linear instability.

We use the velocity scaling

$$u = UA/(A + x), \qquad v = VA/(A + x), \qquad u_\infty = U_\infty A/(A + x) \tag{4}$$

to convert Eqs. (1) and (2) to

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{5}$$

$$U\frac{\partial U}{\partial x} - \frac{U^2}{A + x} + V\frac{\partial U}{\partial y} = U_\infty \frac{\mathrm{d}U_\infty}{\mathrm{d}x} - \frac{U_\infty^2}{A + x} + \nu\left[\frac{A + x}{A}\right]\frac{\partial^2 U}{\partial y^2} \tag{6}$$

respectively. The boundary conditions are

$$U = V = 0, \qquad \text{at } y = 0, \qquad U \to U_\infty, \qquad \text{as } y \to \infty \tag{7}$$

It can be seen that continuity equation (5) is two dimensional, but momentum equation (6) now has two additional terms, compared

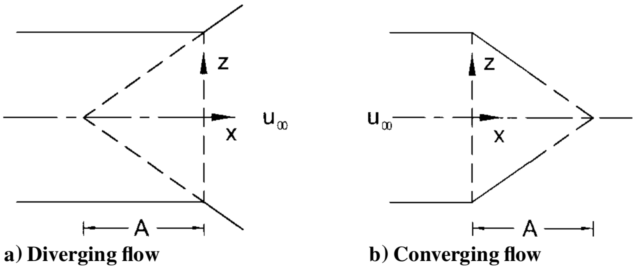

Fig. 1 Flow geometry.

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to the two-dimensional case. These two terms have arisen due to the velocity scaling (4) and can be interpreted as follows. Since $(A+x)>0$ for a diverging flow, there is a reduction in the flow inertia. This reduction in inertia, however, is compensated by an increase in the viscous diffusion rate by a factor $(A+x)/A$, which is the nondimensional divergence/convergence factor. An increased viscous diffusion implies a thinner boundary layer, as observed by Ramesh[4] for laminar flows. Similarly, because $(A+x)<0$ for a converging flow, an enhanced flow inertia in this case is compensated by a reduction in the viscous diffusion rate by a factor $(A+x)/A$. A reduced diffusion in this case implies a thicker boundary-layer thickness, as observed by Vasudevan[5] for laminar flows. In constant pressure laterally strained turbulent flows, thinning/thickening of the boundary layer has been observed by Saddoughi and Joubert[2] and Panchapakesan et al.[3]

When the similarity variables are used,

$$\eta = y\left[\frac{U_\infty(1+m)}{\nu(A+x)}\right]^{\frac{1}{2}}\left[\frac{A}{A+x}\right]^{\frac{1}{2}}$$

$$\psi = \left[\frac{U_\infty \nu(A+x)}{(1+m)}\right]^{\frac{1}{2}}\left[\frac{A+x}{A}\right]^{\frac{1}{2}} F(\eta) \qquad (8)$$

it can easily be verified that Eq. (6) reduces to the Falkner–Skan equation,

$$[(m-1)/(m+1)](F'^2-1) - 0.5FF'' = F''' \qquad (9)$$

under the condition $U_\infty = K(A+x)^m$, where K is a constant. A prime here denotes the derivative with respect to η. The case $m=1$ corresponds to the Blasius equation. Note that this reduction involves a scaling of the velocities by the nondimensional divergence/convergence factor $A/(A+x)$ in Eq. (4). The case when A is very small has to be excluded here, because both u and $v \to 0$ by Eq. (4). A small A implies a very large value of the flow divergence/convergence $[D = 1/(A+x)]$ at the start of the divergence/convergence section (Fig. 1). Thus, the present analysis is valid for small flow divergence/convergence.

B. Linear Instability

Because the mean flow is two dimensional, we consider the linear instability of a two-dimensional flow with a small wavy disturbance of the form $\exp(i\alpha_1 x - i\beta_1 t)$. For the sake of simplicity, we consider the parallel flow instability, although the nonparallel theory is quite well developed now. For the normal mode disturbance, the linear instability equation is the usual Orr–Sommerfeld equation (see Ref. 6),

$$(\Psi''-\alpha^2\Psi)(\alpha F'-\beta)-\alpha F'''\Psi = -i(Re)^{-1}(\Psi''''-2\alpha^2\Psi''+\alpha^4\Psi) \qquad (10)$$

where a prime denotes the derivative with respect to $\eta(=y/\delta^*)$, and $\Psi(\eta)$ is the nondimensional perturbed stream function, $F(\eta)$ is the nondimensional mean flow stream function, Re is the Reynolds number based on the displacement thickness δ^*, and α and β are the nondimensional wave number and frequency, respectively.

Because the mean flow is Blasius, as shown earlier, the neutral solutions of the Orr–Sommerfeld equation (10) will correspond to those for the Blasius flow. [The Orr–Sommerfeld equation (10) has been solved for the temporal mode using the Thomas algorithm,[8] and the eigenvalues are evaluated using the QZ algorithm available in MATLAB®.] The divergence/convergence effect does not show up explicitly. Noting that the reduction to the Blasius flow is via the velocity scaling (4), the divergence/convergence effect on the stability can be seen by scaling the Blasius stability Reynolds numbers by the factor $A/(A+x)$. The neutral curves so obtained are shown in Fig. 2 for two values of $A/(A+x)=1.5$ and 0.75; only two values are considered for the sake of simplicity. It can be seen in Fig. 2 that,

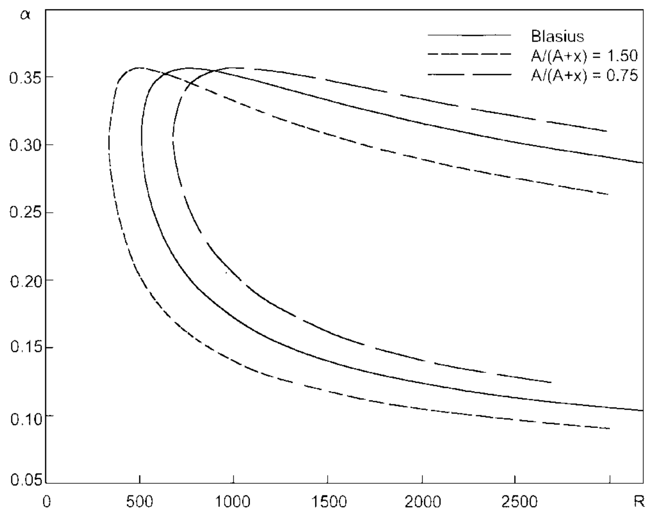

Fig. 2 Neutral curves for diverging and converging flows, along with Blasius flow.

for a given value of $A/(A+x)$, whereas a converging flow is more unstable than the Blasius flow, a diverging flow is more stable. This is expected because the boundary layer is thicker in a converging flow and thinner in a diverging flow, as mentioned earlier. The linear instability analysis of nonzero pressure gradient flows with diverging/converging streamlines merely is an extension of the present analysis because the base flow is reducible to the corresponding Falkner–Skan flow. The present analysis is expected to be useful in practical flows involving similar flow divergence/convergence.

III. Conclusions

The linear instability of a constant pressure boundary-layer flow with diverging/converging streamlines is studied by reducing the flow to the corresponding two-dimensional flow. The instability is found to be that of the Blasius flow at Reynolds numbers scaled by the flow divergence/convergence factor.

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